Cryptanalysis
Part III: Stream Ciphers

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Stream ciphers

- Hardware implementations
- Very fast, energy-efficient

- WEP (Wifi, norme IEEE 802.11, 1999): RC4 (broken)
- Bluetooth: E0
- 2G/3G: A5/1 (backdoored)
- 4G/5G: SNOW

They still exist

- eSTREAM competition (2004-2008) $\implies$ 7 new ciphers, incl. Salsa20
- Chacha20: new variant of Salsa20 with better performance, used in TLS1.3
- GrainAEAD: finalist of the NIST Lightweight standardization (although ASCON won)

LFSR

Feedback Shift Register

Objective: create a pseudorandom sequence that depends only on the initial internal state.

$(s_n)_{n\in\mathbb{N}} \in \mathbb{F}_q^{\mathbb{N}}$ is produced by a feedback function F if there exists $\ell \in \mathbb{N}$ and $F : \mathbb{F}_q^{\ell} \to \mathbb{F}_q$ such that:

$$\forall n \in \mathbb{N}, s_{n+\ell} = F(s_{n+\ell-1}, s_{n+\ell-2}, \dots, s_{n+1}, s_n)$$

Linear Feedback Shift Register

$(s_n)_{n\in\mathbb{N}} \in \mathbb{F}_q^{\mathbb{N}}$ is produced by a **linear feedback function** if there exists $\ell \in \mathbb{N}$ and $F : \mathbb{F}_q^{\ell} \to \mathbb{F}_q$ a **linear form** such that:

$$\forall n \in \mathbb{N}, s_{n+\ell} = F(s_{n+\ell-1}, s_{n+\ell-2}, \ldots, s_{n+1}, s_n)$$

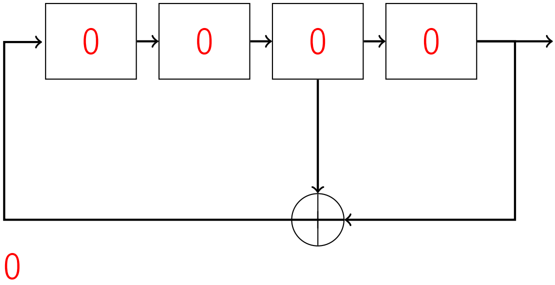
(s_n) is produced by an LFSR if there exists a **constant vector** $(c_1, \ldots, c_\ell) \in \mathbb{F}_q^{\ell}$ such that:

$$\forall n, s_{n+\ell} = c_1 s_{n+\ell-1} + \ldots + c_\ell s_n$$

Every linear recurrent sequence of order ℓ is produced by an LFSR of length ℓ.

Example: binary LFSR

$$\begin{cases} s_{n+4} = s_n + s_{n+1} \\ c_1, c_2, c_3, c_4 = 0, 0, 1, 1 \end{cases} \tag{1}$$



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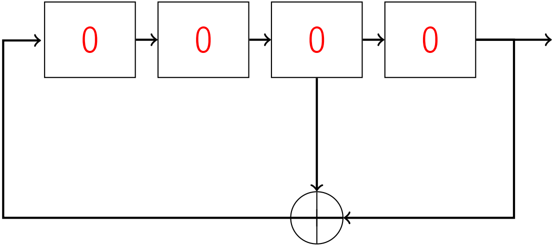

0 0

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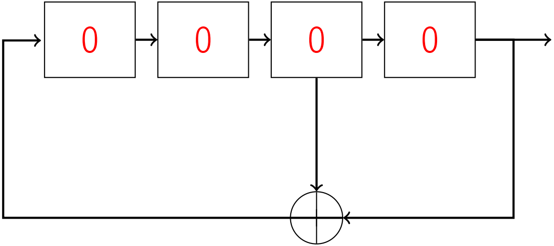
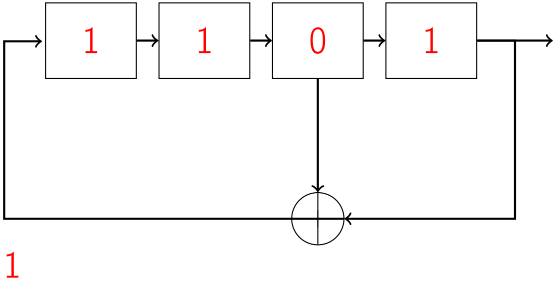


0 0 0 ...

Example: nontrivial binary LFSR



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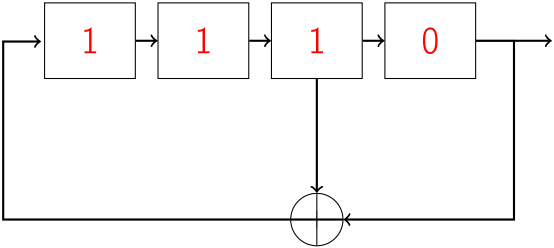

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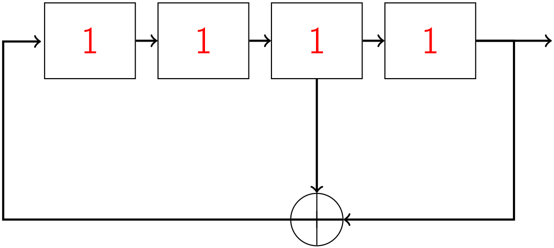
Example: nontrivial binary LFSR



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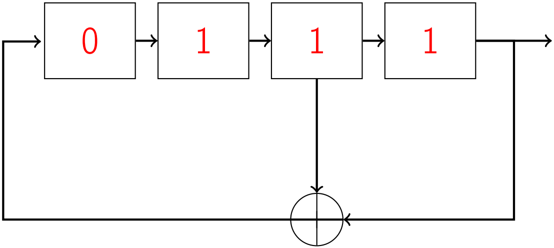
Example: nontrivial binary LFSR



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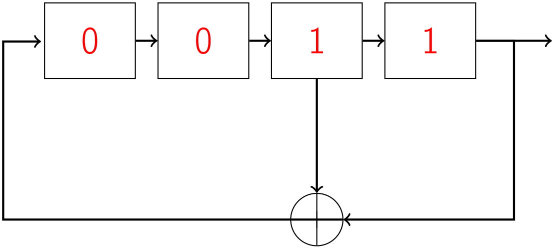

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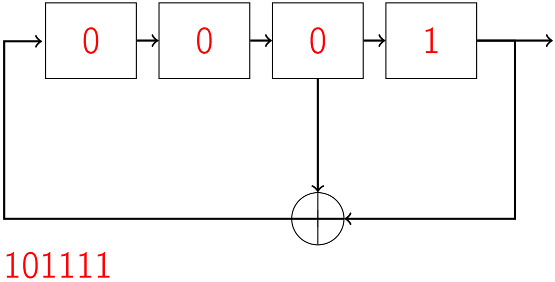
Example: nontrivial binary LFSR



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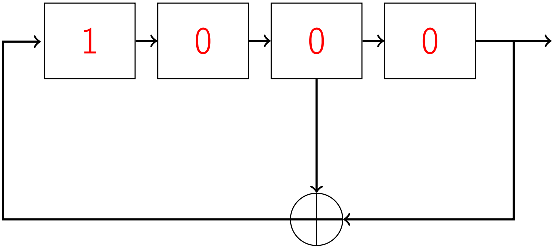

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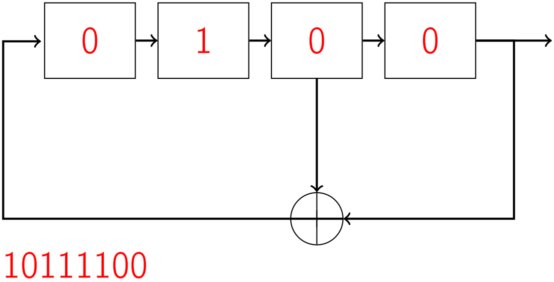
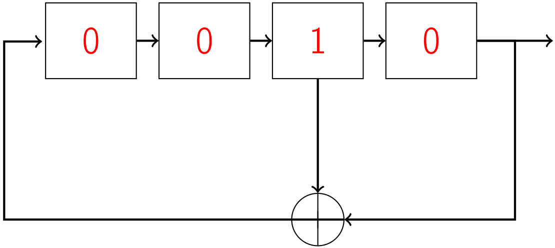
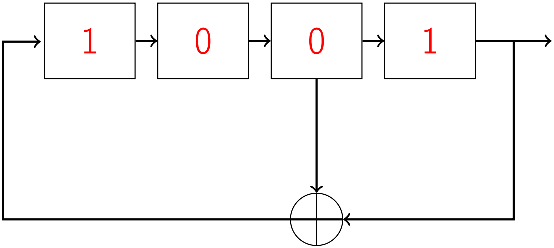


Example: nontrivial binary LFSR



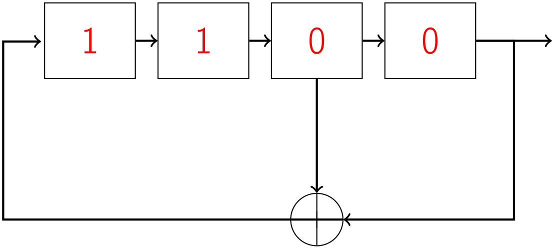
101111000

Example: nontrivial binary LFSR



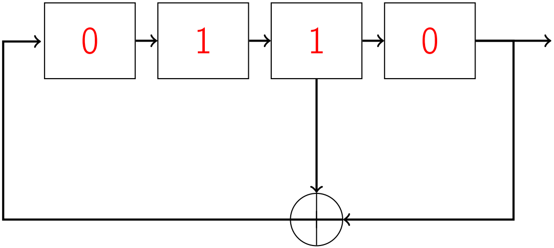
1011110001

Example: nontrivial binary LFSR



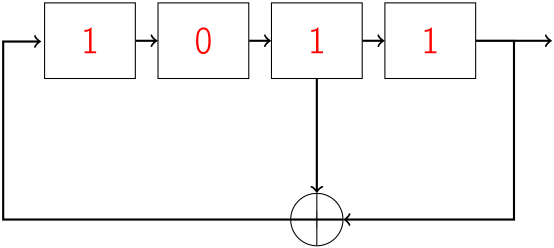
10111100010

Example: nontrivial binary LFSR



101111000100

Example: nontrivial binary LFSR



1011110001001

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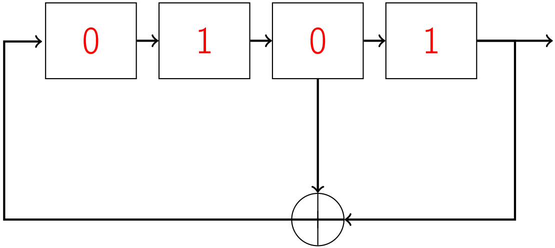

10111100010011

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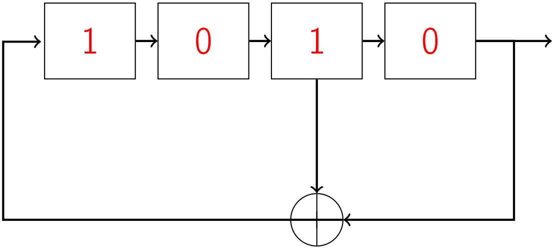

101111000100110 ...

- The period is $15 = 2^4 - 1$
- It is maximal!

Retroaction polynomial

Consider an LFSR of length ℓ and coefficients $c_1, \dots, c_\ell \in \mathbb{F}_q^\ell$. We define the **retroaction polynomial** by:

$$P(X) = 1 - \sum_{i=1}^{\ell} c_i X^i \in \mathbb{F}_q[X]$$

The previous LFSR has the polynomial:

$$P(X) = 1 + X^3 + X^4 \in \mathbb{F}_2[X]$$

$+$ and - in $\mathbb{F}_2$ are the same thing.

Sparsification

Any sequence produced by a LFSR of retroaction P can be produced by any LFSR of retroaction a multiple of P.

Example: let (s_n) be a sequence in $\mathbb{F}_2$ that satisfies:

$$s_{n+6} = s_{n+4} + s_{n+3} + s_{n+1} + s_n, \forall n \geq 6$$

Its retroaction polynomial is $P(X) = 1 + X^2 + X^3 + X^5 + X^6$. The sequence also satisfies: $s_{n+8} = s_{n+7} + s_n$, since:

$$1 + X + X^8 = (1 + X + X^2)P(X)$$

Minimisation

Let $(s_n)_{n \in \mathbb{N}}$ be a linear recurrent sequence.

Among all retroaction polynomials for (s_n), there exists one of **minimal degree**.

Period

Let $(s_n)_{n\in\mathbb{N}}$ be a linear recurrent sequence and P its **minimal** retroaction polynomial. Let ℓ be its degree.

The period of s is $q^\ell - 1$ iff P is **primitive**.

Example: $P(X) = 1 + X^3 + X^4$ is a primitive polynomial; so the period is going to be $2^4 - 1 = 15$.

A primitive LFSR has good statistical properties, but cannot be used alone to construct a stream cipher: we **combine** multiple LFSRs and use a **filtering function**.

Primitive polynomial = monic and one of its roots is a primitive $q^\ell - 1$-root of unity.

The Berlekamp-Massey algorithm

If the minimal polynomial is of degree d, the Berlekamp-Massey algorithm can find it from $2d$ terms of the sequence.

$\Longrightarrow$ useful if we do not know a retroaction polynomial (e.g., for cryptanalysis).

Combining LFSRs

Objective

Combine the outputs of multiple LFSRs:

$$s_t = f(s_t^{(1)}, \dots, s_t^{(n)})$$

where $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is a Boolean function.

- Each LFSR has a primitive retroaction polynomial.
- The characteristics (length, polynomials) are public.
- The initial states of each LFSR (formed from the secret key + IV) are secret.

Interlude: Boolean functions

Definition

A Boolean function in n variables is a function $f : \mathbb{F}_2^n \to \mathbb{F}_2$. It can be described by its truth table.

How many n-variable Boolean functions are there?

Support and weight

- The support of a Boolean function is:

$$Supp(f) = \{\boldsymbol{x} \in \mathbb{F}_2^n, f(\boldsymbol{x}) \neq 0\}$$

- The weight of f is $w(f) = |Supp(f)|$
- f is balanced if $w(f) = 2^{n-1}$

We need the Boolean function to be balanced (otherwise the output will be biased).

Algebraic normal form

There exists a unique multivariate polynomial $\bar{f}$ such that:

$$\bar{f}(X_1, \ldots, X_n) = \sum_{\boldsymbol{u}=(u_{\mathbf{1}},\ldots,u_n)\in\mathbb{F}_{\mathbf{2}}^n} a_{\boldsymbol{u}} X_1^{u_{\mathbf{1}}} \cdots X_n^{u_n}$$

such that: $f(x_1, \ldots, x_n) = \bar{f}(x_1, \ldots, x_n)$

$\bar{f}$ is the **ANF** of f, and:

$$a_{\boldsymbol{u}} = \sum_{x \preceq \boldsymbol{u}} f(x) \text{ where } x \preceq y \text{ iff } x_i \leq y_i \text{ for all } i$$

Remember that this is a sum on $\mathbb{F}_2$.

Algebraic degree

The algebraic complexity of a Boolean function is quantified by the **degree** of its ANF: if

$$f(X_1, \ldots, X_n) = \sum_{\boldsymbol{u} \in \mathbb{F}_2^n} a_{\boldsymbol{u}} X_1^{u_1} \cdots X_n^{u_n}$$

then

$$deg(f) = \max\{hw(\boldsymbol{u}) | a_{\boldsymbol{u}} \neq 0\}$$

where $hw(\boldsymbol{u})$ is the Hamming weight of $\boldsymbol{u}$ (number of ones).
This is the **maximal number of variables in a monomial** of f.

A "random" Boolean function has very large degree ($n-1$). Having small degree is a property that can be used for cryptanalysis.

Example

Geffe proposed to use the function defined by the following truth table to combine 3 LFSRs.

x_1	0	0	0	0	1	1	1	1
x_2	0	0	1	1	0	0	1	1
x_3	0	1	0	1	0	1	0	1
$f(x_1, x_2, x_3)$	0	1	0	0	0	1	1	1

We can see that:

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We can see that:

$$Supp(f) = \{(0,0,1),(1,0,1),(1,1,0),(1,1,1)\}$$

and $w(f) = 4$.

ANF

$$
\begin{aligned}
a_{000} &= f(0,0,0) = 0 \\
a_{001} &= f(0,0,0) + f(0,0,1) = 1 \\
a_{010} &= f(0,0,0) + f(0,1,0) = 0 \\
a_{011} &= f(0,0,0) + f(0,1,0) + f(0,0,1) + f(0,1,1) = 1 \\
a_{100} &= f(0,0,0) + f(1,0,0) = 0 \\
a_{101} &= f(0,0,0) + f(1,0,0) + f(0,0,1) + f(1,0,1) = 0 \\
a_{110} &= f(0,0,0) + f(1,0,0) + f(0,1,0) + f(1,1,0) = 1 \\
a_{111} &= \sum_{\boldsymbol{u}} f(\boldsymbol{u}) = w(f) \mod 2 = 0
\end{aligned}
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ANF

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a_{100} &= f(0,0,0) + f(1,0,0) = 0 \\
a_{101} &= f(0,0,0) + f(1,0,0) + f(0,0,1) + f(1,0,1) = 0 \\
a_{110} &= f(0,0,0) + f(1,0,0) + f(0,1,0) + f(1,1,0) = 1 \\
a_{111} &= \sum_{\boldsymbol{u}} f(\boldsymbol{u}) = w(f) \mod 2 = 0
\end{aligned}
$$

$f(X_1, X_2, X_3) = X_3 + X_2X_3 + X_1X_2$ and $deg(f) = 2$

Linear complexity of the combined sequence

The degree of the minimal polynomial is named **linear complexity** of the sequence and noted $\Lambda(s)$.

Lemma (Rueppel, Staffelbach, 1987)

Let s^1 and s^2 be two linear recurrent sequences, of minimal polynomials P^1 and P^2. Then:

- $\Lambda(s^1 + s^2) \leq \Lambda(s^1) + \Lambda(s^2)$ with equality iff $gcd(P^1, P^2) = 1$
- $\Lambda(s^1 * s^2) \leq \Lambda(s^1)\Lambda(s^2)$. If P^1, P^2 are primitive, of degrees distinct and bigger than 2, there is an equality.

Here $s^1 * s^2$ means the pointwise product of the sequences.

Linear complexity of the combined sequence

Corollary

Let $s^1, \ldots, s^n$ be linear recurrent sequences produced by LFSRs of respective lengths $\ell^1, \ldots, \ell^n$. Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function. The combined sequence $f(s^1, \ldots, s^n)$ has linear complexity:

$$\Lambda = f(\ell^1, \ldots, \ell^n)$$

obtained by evaluating the ANF of f as a polynomial in $\mathbb{Z}$.

Example: for Geffe's cipher: $f(x_1, x_2, x_3) = x_3 + x_2x_3 + x_1x_2$
The linear complexity is $\Lambda = \ell^3 + \ell^2\ell^3 + \ell^1\ell^2$.

Filtered LFSRs

- Take several bits of the same LFSR
- Equivalent: Combine k LFSRs with the same retroaction polynomial, but shifted initial states
- Caution: previous results do not apply

(Edwin L. Key, 1976)

The linear complexity $\Lambda(s)$ of a sequence s produced by an LFSR of length ℓ and filtered by a Boolean function of degree d satisfies:

$$\Lambda(s) \leq \sum_{i=0}^{d} \binom{\ell}{i}$$

(Rueppel, 1986)

When ℓ is prime and big enough, then $\Lambda(s) \simeq \binom{\ell}{d}$ for most degree-d Boolean functions.

Cryptanalysis

Correlation attacks

- Consider n LFSRs of lengths $\ell_1, \ldots, \ell_n$ with a post-processing function f.
- **Goal:** find the internal states.

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- **Goal:** find the internal states.

Exhaustive search: $= \prod_{i=1}^{n}(2^{\ell_i} - 1)$

Correlation attack: principle

If f is **badly chosen**, the output sequence may be correlated to a sequence formed by **less LFSRs**.

- Perform an exhaustive search of the internal states of these LFSRs
- Check if the output sequence is correlated as we expect
- Once the internal state of an LFSR is obtained, continue with the others

e.g., $\prod_i (2^{\ell_i} - 1) \to \sum_i (2^{\ell_i} - 1)$

Countermeasures

A boolean function f is **uncorrelated to order k** if for all independent random variables $X_1, \ldots, X_n$, the random variable $f(X_1, \ldots, X_n)$ is independent from any $(X_{i_1}, \ldots, X_{i_k})$. The largest such k is the **immunity** of f to correlations.

We need to choose f with a strong immunity, and a large algebraic degree.